

Integration on differential spaces

Diana Dziewa-Dawidczyk* Zbigniew Pasternak-Winiarski†

January 1, 2013

Abstract

The generalization of the n -dimensional cube, an n -dimensional chain, the exterior derivative and the integral of a differential n -form on it are introduced and investigated. The analogue of Stokes theorem for the differential space is given.

Key words and phrases: differential space, integration.

2000 AMS Subject Classification Code 58A40, 26A18.

1 Introduction

This article is the fifth of the series of papers concerning integration of differential forms and densities on differential spaces. We describe our generalization of the theory of integration of smooth skew-symmetric forms on cubes and chains.

In Section 2 we present definition and properties of so called *point differential forms* on differential spaces. We prove the theorem about the local representation of such forms (Theorem 1). Section 3 of the paper contains basic definitions and the description of preliminary facts concerning generalized cubes and chains. We define the notion of a *generalized n -dimensional smooth cube* on the differential space (M, \mathcal{C}) , the notion of a *smooth n -dimensional chain* of generalized smooth cubes on (M, \mathcal{C}) and the integral of a smooth point n -form on a smooth n -dimensional chain. In Section 4 we prove the existence of integrals for a wide class of smooth chains and skew-symmetric forms. To formulate this results we

*Department of Applied Mathematics, Faculty of Applied Informatics and Mathematics, Warsaw University of Life Sciences - SGGW, Ul. Nowoursynowska 159, 02-776 Warsaw, POLAND, e-mail: diana_dziewa_dawidczyk@sggw.pl

†Faculty of Mathematics and Information Science, Warsaw University of Technology, Ul. Koszykowa 75, 00-662 Warsaw, POLAND, Institute of Mathematics, University of Białystok Akademicka 2, 15-267 Białystok, Poland, e-mail: Z.Pasternak-Winiarski@mini.pw.edu.pl

introduce the classes of cubes, chains and point forms *smoothly extendable with respect to the family \mathcal{G}* of generators of a differential structure \mathcal{C} on a set M . At the end of the paper we give an analogue of Stokes theorem in the case of smoothly extendable forms and chains (Theorem 3). To do that we introduce *the boundary of generalized n -dimensional chain* (Definition 13).

Without any other explanation we use the following symbols: \mathbb{N} - the set of natural numbers; \mathbb{Q} - the set of rational numbers; \mathbb{R} - the set of reals.

2 Point forms

Let (M, \mathcal{C}) be a differential space and let $TM = \bigsqcup_{m \in M} T_m M$ be the space tangent to (M, \mathcal{C}) (the disjoint union of point tangent spaces). Let $\pi : TM \rightarrow M$ be the natural projection, i.e. $\pi(v) = m$ for any $v \in T_m M$. We endow TM with the differential structure \mathcal{TC} generated by the family of function $\{\alpha \circ \pi, \alpha \in \mathcal{C}\} \cup \{d\alpha, \alpha \in \mathcal{C}\}$, where $d\alpha : TM \rightarrow \mathbb{R}$, $d\alpha(v) = v(\alpha)$ (see [2], Definition 2.4 and remarks after this definition). Then the map π is smooth with respect to differential structures \mathcal{C} and \mathcal{TC} .

Definition 1. A map $\omega : TM \rightarrow \mathbb{R}$ is called a *point 1-form* if for any $m \in M$ $\omega_m = \omega|_{T_m M}$ is an \mathbb{R} -linear mapping. A point 1-form ω is said to be *smooth* if $\omega \in \mathcal{TC}$ (see [7]).

Let us take the notation:

$$T^k M = \{(v_1, \dots, v_k) \in TM \times \dots \times TM : \pi(v_1) = \dots = \pi(v_k)\},$$

$$\tau^k \mathcal{C} = (\mathcal{TC} \hat{\otimes} \dots \hat{\otimes} \mathcal{TC})_{T^k M} \text{ for } k = 1, 2, \dots$$

(see [2], remarks after Proposition 2.2).

Let $\pi_i : T^k M \rightarrow TM$ for $i = 1, \dots, k$ be a map given by the formula

$$\pi_i(v_1, \dots, v_k) = v_i, \quad (v_1, \dots, v_k) \in T^k M.$$

Definition 2. A map $\omega : T^k M \rightarrow \mathbb{R}$ is called a *point k -form* if for any $m \in M$ $\omega_m = \omega|_{T_m M}$ is a k -linear mapping. A point k -form ω is said to be *smooth* if $\omega \in \tau^k \mathcal{C}$ (see [7]).

Lemma 3. Let W be an open set in \mathbb{R}^m , U – an open neighborhood of zero in \mathbb{R}^n and $\sigma : W \times U \rightarrow \mathbb{R}$ – a C^∞ -function. If for a point $(w, u) = (w_1, \dots, w_m, u_1, \dots, u_n) \in W \times U$ there exist $a < 0$ and $b > 1$ such that for any $t \in (a; b)$ the point $(w, tu) \in W \times U$ and $\sigma(w, tu) = t\sigma(w, u)$ then

$$\sigma(w, u) = \sum_{i=1}^n \frac{\partial \sigma}{\partial u_i}(w, 0) \cdot u_i.$$

Proof. Since the map $(a; b) \ni t \mapsto \sigma(w, tu) = t\sigma(w, u) \in \mathbb{R}$ is well defined and smooth on (a, b) we have

$$\begin{aligned}\sigma(w, u) &= \frac{d}{dt}(t\sigma(w, u)|_{t=0}) = \frac{d}{dt}(\sigma(w, tu)|_{t=0}) = \sum_{i=1}^n \frac{\partial \sigma}{\partial u_i}(w, 0) \cdot \frac{d(tu_i)}{dt} \\ &= \sum_{i=1}^n \frac{\partial \sigma}{\partial u_i}(w, 0) \cdot u_i.\end{aligned}$$

□

Using induction we can extend this result to the following one.

Lemma 4. *Let W be an open set in \mathbb{R}^m and let for any $j = 1, 2, \dots, k$ a set U^j be an open neighborhood of zero in \mathbb{R}^{n_j} ($n_1, \dots, n_k \in \mathbb{N}$). Suppose that $\sigma : W \times U^1 \times \dots \times U^k \rightarrow \mathbb{R}$ is a C^∞ -function, $(w, u^1, \dots, u^k) \in W \times U^1 \times \dots \times U^k$ and there exist $a < 0$ and $b > 1$ such that for any $t_1, \dots, t_k \in (a; b)$ we have $(w, t_1 u^1, \dots, t_k u^k) \in W \times U^1 \times \dots \times U^k$ and*

$$\sigma(w, t_1 u^1, \dots, t_k u^k) = t_1 t_2 \dots t_k \sigma(w, u^1, \dots, u^k).$$

Then

$$\sigma(w, u^1, \dots, u^k) = \sum_{j_1=1}^{n_1} \dots \sum_{j_k=1}^{n_k} \frac{\partial^k \sigma}{\partial u_{j_1}^1 \dots \partial u_{j_k}^k}(w, 0, \dots, 0) u_{j_1}^1 \dots u_{j_k}^k.$$

□

Theorem 1. *If $\omega : T^k M \rightarrow \mathbb{R}$ is a smooth point k -form on a differential space (M, \mathcal{C}) then for any $m \in M$ there exist a neighborhood W_m of m , a number $n \in \mathbb{N}$, functions $\alpha_1, \dots, \alpha_n \in \mathcal{C}$ and for any $i_1, \dots, i_k \in \{1, 2, \dots, n\}$ a function $\omega_{i_1 \dots i_k} \in \mathcal{C}$ such that*

$$\omega|_{T^k W_m} = \sum_{i_1 \dots i_k=1}^n [(\omega_{i_1 \dots i_k} \circ \pi \circ \pi_1) d\alpha_{i_1} \otimes \dots \otimes d\alpha_{i_k}]|_{T^k W_m}, \quad (1)$$

where $T^k W_m = \bigsqcup_{p \in W_m} T_m^k W_m$, $\pi : TM \rightarrow M$ is the natural projection and $\pi_i : T^k M \rightarrow TM$ is described after Definition 1.

Remark 5. *Taking into account that functions $\omega_{i_1 \dots i_k}$ and α_j depend only on $p \in W_m$ we will abbreviate formula (1) writing*

$$\omega = \sum_{i_1 \dots i_k=1}^n \omega_{i_1 \dots i_k} d\alpha_{i_1} \otimes \dots \otimes d\alpha_{i_k} \quad \text{on } W_m. \quad (2)$$

Proof of the theorem. It follows from the definition of the differential structure $\tau^k \mathcal{C}$ (see [2]) that there exist: a neighborhood \tilde{V} of the point $(0_m, \dots, 0_m) \in T_m^k M$ in $T^k M$ (where 0_m is the neutral element of the vector space T_m), functions $\beta_1^0, \dots, \beta_{n_0}^0, \beta_1^1, \dots, \beta_{n_1}^1, \dots, \beta_1^k, \dots, \beta_{n_k}^k \in \mathcal{C}$ and a function $\sigma \in C^\infty(\mathbb{R}^n)$, where $n = n_0 + n_1 + \dots + n_k$, such that for any $(v_1, \dots, v_k) \in \tilde{V}$

$$\omega(v_1, \dots, v_k) = \sigma(\beta_1^0(\pi(v_1)), \dots, \beta_{n_0}^0(\pi(v_1)), d\beta_1^1(v_1), \dots, d\beta_{n_1}^1(v_1), \dots, d\beta_1^k(v_k), \dots, d\beta_{n_k}^k(v_k)).$$

Reducing \tilde{V} , if necessary, we can assume that $\tilde{V} = \beta^{-1}(U^0 \times U^1 \times \dots \times U^k)$, where $\beta : T^k M \rightarrow \mathbb{R}^n$ is given by the formula

$$\begin{aligned} \beta(v_1, \dots, v_k) = & (\beta_1^0(\pi(v_1)), \dots, \beta_{n_0}^0(\pi(v_1)), d\beta_1^1(v_1), \dots, d\beta_{n_1}^1(v_1), \dots, d\beta_1^k(v_k), \dots, d\beta_{n_k}^k(v_k)), \\ & (v_1, \dots, v_k) \in T^k M \end{aligned}$$

and $U^j \subset \mathbb{R}^{n_j}$ is an open n_j -dimensional interval (cube) for $j = 0, 1, \dots, k$ (see [1], Stwierdzenie 2.7 or [8]). Note that if $(t_1, \dots, t_k) \in \mathbb{R}^k$ and $(v_1, \dots, v_k) \in \tilde{V}$ fulfill the condition

$$(t_1 d\beta_1^1(v_1), \dots, t_1 d\beta_{n_1}^1(v_1), \dots, t_k d\beta_1^k(v_k), \dots, d\beta_{n_k}^k(v_k)) \in U^1 \times \dots \times U^k$$

then

$$(t_1 v^1, t_2 v^2, \dots, t_k v^k) \in \tilde{V}.$$

Taking $W := (\pi \circ \pi_1)(\tilde{V})$, $w := \beta^0(\pi(v_1)) := (\beta_1^0(\pi(v_1)), \dots, \beta_{n_0}^0(\pi(v_1)))$ and $u^j := (d\beta_1^j(v_j), \dots, d\beta_{n_j}^j(v_j)) \in \mathbb{R}^{n_j}$ for $j = 1, \dots, k$ we obtain that the sets W, U^1, \dots, U^k , the point (w, u^1, \dots, u^k) and the map σ fulfill conditions of Lemma 4. By the thesis of this lemma

$$\begin{aligned} \omega(v_1, \dots, v_k) &= \sum_{j_1=1}^{n_1} \dots \sum_{j_k=1}^{n_k} \frac{\partial^k \sigma}{\partial u_{j_1}^1 \dots \partial u_{j_k}^k} (w, 0, \dots, 0) d\beta_{j_1}^1(v_1) \dots d\beta_{j_k}^k(v_k) \\ &= \sum_{j_1=1}^{n_1} \dots \sum_{j_k=1}^{n_k} \frac{\partial^k \sigma}{\partial u_{j_1}^1 \dots \partial u_{j_k}^k} (\beta^0(\pi(v_1)), 0, \dots, 0) d\beta_{j_1}^1 \otimes \dots \otimes d\beta_{j_k}^k(v_1, \dots, v_k). \end{aligned} \tag{3}$$

Note that the function $T^k M \ni (v_1, \dots, v_k) \mapsto \frac{\partial^k \sigma}{\partial u_{j_1}^1 \dots \partial u_{j_k}^k} (\beta^0(\pi(v_1)), 0, \dots, 0) \in \mathbb{R}$ belongs to \mathcal{C} . Putting $\alpha_i := \beta_i^0$ for $i = 1, 2, \dots, n_0$, $\alpha_{n_0+n_1+\dots+n_l+i} := \beta_i^{l+1}$ for $l = 0, 1, \dots, k-1$ and $i = 1, \dots, n_{l+1}$, $n := n_0 + n_1 + \dots + n_k$, $\omega_{i_1 \dots i_n}(q) :=$

$\frac{\partial^k \sigma}{\partial u_{j_1}^1 \dots \partial u_{j_k}^k}(\beta^0(q), 0, \dots, 0)$ if $d\alpha_{i_1} \otimes \dots \otimes d\alpha_{i_k} = d\beta_{j_1}^1 \otimes \dots \otimes d\beta_{j_k}^k$ ($q \in W$) and $\omega_{i_1 \dots i_n}(q) := 0$ if $d\alpha_{i_1} \otimes \dots \otimes d\alpha_{i_k}$ does not appear in (3), we obtain that (1) is true if we change $T^k W_m$ onto \tilde{V} . Hence, by k -linearity of ω , it follows that (1) holds for $T^k W_m$, where $W_m = W$. \square

For any smooth map F of a differential space (M, \mathcal{C}) into a differential space (N, \mathcal{D}) (we denote it writing $F : (M, \mathcal{C}) \rightarrow (N, \mathcal{D})$) and any smooth k -form ω on (N, \mathcal{D}) one can define the smooth k -form $F^* \omega$ by the formula:

$$(F^* \omega)(v_1, \dots, v_k) = \omega(TFv_1, \dots, TFv_k), \quad (v_1, \dots, v_k) \in T^k M,$$

where the tangent map $TF : TM \rightarrow TN$ is defined as follows

$$(TFv)(f) = v(f \circ F), \quad v \in TM, \quad f \in \mathcal{D}.$$

Note that for any $m \in M$ the map $TF|_{T_m M}$ is linear.

Proposition 1. *If we denote $F = (\alpha_0, \dots, \alpha_n) : W_m \rightarrow \mathbb{R}^n$ in the proof of Theorem 1 then the equality (2) can be written in the form*

$$\omega|_{W_m} = (F^* \eta)|_{W_m}$$

(here we consider ω as a section of the k -power of cotangent bundle on M), where $\eta = \sum_{i_1 \dots i_k=1}^n \eta_{i_1 \dots i_k} du_{i_1} \otimes \dots \otimes du_{i_k}$ is the smooth k -linear form on \mathbb{R}^n given as follows

$$\eta_{i_1 \dots i_k}(u) = \frac{\partial^k \sigma}{\partial u_{n_0+i_1} \dots \partial u_{n_0+n_1+\dots+n_{k-1}+i_k}}(u_1, \dots, u_{n_0}, 0, \dots, 0)$$

if $d\alpha_{i_1} \otimes \dots \otimes d\alpha_{i_k} = d\beta_{j_1}^1 \otimes \dots \otimes d\beta_{j_k}^k$ for some $1 \leq j_1 \leq n_1, \dots, 1 \leq j_k \leq n_k$ and

$$\eta_{i_1 \dots i_k}(u) = 0$$

in other cases.

Proof. It follows from the fact that for any $n_0 < i < n$, any $q \in W_m$ and any $v \in T_q M$

$$F^* du_i(v) = (TFv)(u_i) = v(u_i \circ F) = v(\alpha_i) = d\alpha_i(v).$$

\square

Remark 6. *It is easy to see that if \mathcal{G} is an arbitrary family of generators of the differential structure \mathcal{C} then in the formulation of Theorem 1 and Proposition 1 we can put \mathcal{G} instead of \mathcal{C} preserving only the condition: $\omega_{i_1 \dots i_k} \in \mathcal{C}$. For the definition and basic properties of a set of generators of a differential structure see [1], [2], [3] or [4].*

In the following part of the paper we will be interested only in skew-symmetric k -forms. In this case equalities (1) and (2) take the form

$$\omega = \sum_{1 \leq i_1 < \dots < i_k \leq n} \omega_{i_1 \dots i_k} d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_k} \quad \text{on } W_m. \quad (4)$$

3 Definition of the integral on generalized cubes and chains

Definition 7. The Cartesian product $[0, 1]^n$ of n unit intervals is called *the standard n -dimensional interval*. A continuous mapping $\varphi : [0, 1]^n \rightarrow A$ is said to be *n -dimensional cube* in a topological space A . If (M, \mathcal{C}) is a differential space, then the cube $\varphi : [0, 1]^n \rightarrow M$ is *smooth* when $\varphi : ([0, 1]^n, C^\infty([0, 1]^n)) \rightarrow (M, \mathcal{C})$, i.e. φ is a smooth mapping on the differential space $([0, 1]^n, C^\infty([0, 1]^n))$ into the differential space (M, \mathcal{C}) . The integral of a smooth point n -form ω (on M) on a smooth cube φ is defined as the number

$$\int_{\varphi} \omega := \int_{[0, 1]^n} \varphi^* \omega,$$

where $\varphi^* \omega$ is the pullback of the form ω onto the interval $[0, 1]^n$.

Example 1. Let $M = \mathbb{Q}$ be the set of rational numbers and $\mathcal{C} := C^\infty(\mathbb{R})_{\mathbb{Q}}$. Then for any number $n \in \mathbb{N}$ only constant functions are n -dimensional cubes in M . If φ is a constant cube (of any dimension) and ω is a point form on \mathbb{Q} (the same dimension), then $\varphi^* \omega \equiv 0$. So we have $\int_{\varphi} \omega = 0$. \square

It follows from the example above that the standard integration theory might be trivial on some differential spaces. In fact, on the space $(\mathbb{Q}, C^\infty(\mathbb{R})_{\mathbb{Q}})$ there are non trivial point 1-forms, such as $\omega = d(id_{\mathbb{Q}})$, but all integrals of such forms vanish. We would like to generalize integration theory in a way, which enables us to omit this inconvenience. We have to give the generalization of n -dimensional cubes and chains. In order to do this we will use the theory of completions and compactifications of a differential space described in [2]. At first we will extend smooth functions.

Let (M, \mathcal{C}) be a differential space such that M is Hausdorff space and \mathcal{C} is generated by the family of functions \mathcal{G} , which defines uniform structure $\mathcal{U}_{\mathcal{G}}$ on M (see [2]). Any function $f \in \mathcal{C}$ which is uniform with respect to $\mathcal{U}_{\mathcal{G}}$ and standard uniform structure on \mathbb{R} can be extended in the univocal way to the continuous function on the completion $compl_{\mathcal{G}} M$ of the uniform space $(M, \mathcal{U}_{\mathcal{G}})$. In particular

any function $g \in \mathcal{G}$ can be extended to the continuous function \tilde{g} on $\text{compl}_{\mathcal{G}}M$. The localization of the differential structure $\text{compl}_{\mathcal{G}}\mathcal{C}$ generated by the family $\tilde{\mathcal{G}} = \{\tilde{g} : \text{compl}_{\mathcal{G}}M \rightarrow \mathbb{R} : g \in \mathcal{G}\}$ to the set M is equal to \mathcal{C} . If $\mathcal{G} = \mathcal{C}$, then any function $\alpha \in \mathcal{C}$ can be extended to $\tilde{\alpha} \in \text{compl}_{\mathcal{C}}\mathcal{C}$ on the maximal differential completion $\text{compl}_{\mathcal{C}}M$.

Definition 8. A generalized n -dimensional smooth cube on the differential space (M, \mathcal{C}) is any smooth mapping $\varphi : (D, C^\infty(\mathbb{R}^n)_D) \rightarrow (M, \mathcal{C})$ defined on the set D dense in $[0; 1]^n$.

If \mathcal{G} is such a set of the generators of the structure \mathcal{C} , that φ is uniform with respect to uniform structure $\mathcal{U}_{\mathcal{G}}$ on M (and standard uniform structure on D), then φ can be extended to n -dimensional cube $\tilde{\varphi} : [0; 1]^n \rightarrow \text{compl}_{\mathcal{G}}M$ (not necessarily smooth with respect to $\text{compl}_{\mathcal{G}}\mathcal{C}$).

Example 2. Let M be the set of all square roots of rational numbers belonging to the interval $(0; 1)$. A mapping

$$\varphi(x) = \sqrt{x}; \quad x \in (0; 1) \cap \mathbb{Q},$$

is a generalized smooth 1-dimensional cube in the space $(M, C^\infty(\mathbb{R})_M)$ and it is uniform with respect to the uniform structure $\mathcal{U}_{\mathcal{G}}$ defined by $\mathcal{G} = \{id_M\}$. But its extension $\tilde{\varphi} : [0; 1] \rightarrow \text{compl}_{\mathcal{G}}M = [0; 1]$,

$$\tilde{\varphi}(x) = \sqrt{x}, \quad x \in [0; 1],$$

is not smooth with respect to $\text{compl}_{\mathcal{G}}\mathcal{C} = C^\infty([0; 1])$. \square

If $\varphi : D \rightarrow M$ is a generalized n -dimensional smooth cube on the differential space (M, \mathcal{C}) , and ω is a smooth point n -form on M , then the pullback $\varphi^*\omega$ is a smooth point n -form on D . That form is represented by the formula:

$$[\varphi^*\omega](t) = \omega_0(t)dt^1 \wedge \dots \wedge dt^n, \quad t = (t^1, \dots, t^n) \in D,$$

where ω_0 is a smooth function on D . If the function ω_0 could be extended to a continuous function $\tilde{\omega}_0 : [0, 1]^n \rightarrow \mathbb{R}$ (the function $\tilde{\omega}_0$ is given univocally because D is a dense set in $[0, 1]^n$), then we call the form

$$\widetilde{\varphi^*\omega} := \tilde{\omega}_0 dt^1 \wedge \dots \wedge dt^n$$

the continuous extension of $\varphi^*\omega$ on the cube $[0, 1]^n$. In that case we can define the integral.

Definition 9. Let ω be a smooth point n -form on the differential space (M, \mathcal{C}) . The integral of ω on the generalized n -dimensional smooth cube $\varphi : D \rightarrow M$ is the number

$$\int_{\varphi} \omega := \int_{[0,1]^n} \widetilde{\varphi^* \omega},$$

where $\widetilde{\varphi^* \omega}$ is the continuous extension of n -form $\varphi^* \omega$ on the cube $[0, 1]^n$ (if the extension exists).

Similarly as in the case of differential manifolds, we can treat the integral of the smooth point form on the generalized n -dimensional cube φ as a value of the mapping given on the set of such generalized smooth cubes, that the integral exists. We can extend that mapping on the set of n -dimensional chains on the differential space (M, \mathcal{C}) .

Definition 10. Let $Cube_n(M, \mathcal{C})$ be the set of all n -dimensional smooth generalized cubes on a differential space (M, \mathcal{C}) . A smooth n -dimensional chain on (M, \mathcal{C}) is any function $\Phi : Cube_n(M, \mathcal{C}) \rightarrow \mathbb{R}$ such that its values are different from zero on at most finite subset of a set $Cube_n(M, \mathcal{C})$. We denote the set of all smooth n -dimensional chains on (M, \mathcal{C}) by $Ch_n(M, \mathcal{C})$.

The set $Ch_n(M, \mathcal{C})$ has the natural structure of a vector space over \mathbb{R} , as a set of real-valued functions on $Cube_n(M, \mathcal{C})$. "Zero chain" is the function equal to zero at each cube. We identify any element φ of a set $Cube_n(M, \mathcal{C})$ with a chain $\Phi \in Ch_n(M, \mathcal{C})$ such that $\Phi(\varphi) = 1$ and $\Phi(\psi) = 0$ for $\psi \in Cube_n(M, \mathcal{C}) \setminus \{\varphi\}$. We denote that chain by $1 \cdot \varphi$.

If $\Phi \in Ch_n(M, \mathcal{C})$, then we define the support of Φ as the set $supp\Phi = \{\varphi \in Cube_n(M, \mathcal{C}) : \Phi(\varphi) \neq 0\}$. From the definition of a chain we get that the set $supp\Phi$ is finite. If $supp\Phi = \{\varphi_k, \dots, \varphi_m\}$, where $k, m \in \mathbb{N}$, $k \leq m$ and $\Phi(\varphi_j) = c_j$ for $j = k, \dots, m$, then the chain Φ can be represented by the formula

$$\Phi = \sum_{j=k}^m c_j \varphi_j. \quad (5)$$

If $\Phi' = \sum_{j=1}^m c_j \varphi_j$, $\Phi'' = \sum_{j=m+1}^n c_j \varphi_j$, then $\Phi' + \Phi'' = \sum_{j=1}^n c_j \varphi_j$. Assuming that the input of zero components into the sum (5) is zero and taking $c_{\varphi} := \Phi(\varphi)$ we can write $\Phi = \sum_{\varphi \in Cube_n(M, \mathcal{C})} c_{\varphi} \varphi$.

If ω is a smooth n -dimensional point form on (M, \mathcal{C}) , then by $Cube_n^{\omega}(M, \mathcal{C})$ we denote the set of all $\varphi \in Cube_n(M, \mathcal{C})$, such that there exists continuous extension $\widetilde{\varphi^* \omega}$ of the form $\varphi^* \omega$ onto $[0, 1]^n$. Similarly, by $Ch_n^{\omega}(M, \mathcal{C})$ we denote the set of all chains $\Phi \in Ch_n(M, \mathcal{C})$, such that $supp\Phi \subset Cube_n^{\omega}(M, \mathcal{C})$.

Definition 11. *The integral of a smooth point n -form ω on (M, \mathcal{C}) on a smooth chain $\Phi = \sum_{j=1}^n c_j \varphi_j \in Ch_n^\omega(M, \mathcal{C})$ is the number*

$$\int_{\Phi} \omega = \sum_{j=1}^n c_j \int_{\varphi_j} \omega.$$

The mapping $Ch_n^\omega(M, \mathcal{C}) \ni \Phi \mapsto \int_{\Phi} \omega \in \mathbb{R}$ is, of course, a linear functional on $Ch_n^\omega(M, \mathcal{C})$. We will call the mapping "integral of the n -form ω ", too.

4 The existence of integrals and Stokes theorem

The problem is that the domain $Ch_n^\omega(M, \mathcal{C})$ of the integral depends on the form ω (in the case of integrals of smooth n -forms on ordinary chains on a smooth manifold we can integrate each smooth n -form on each chain of class C^1). In the first part of this section we will give sufficient conditions for existence of the continuous or smooth extension $\widetilde{\varphi^* \omega}$ of a point n -form $\varphi^* \omega$ from the domain of the mapping φ to $[0, 1]^n$, where ω is a smooth point n -form and φ is a generalized smooth n -dimensional cube on the differential space (M, \mathcal{C}) .

If D is a dense subset in $[0, 1]^n$, then the space $T_t D$ tangent to the differential space $(D, C^\infty(\mathbb{R}^n))$ at the point $t = (t^1, \dots, t^n) \in D$ is an n -dimensional vector space which can be identified with $T_t \mathbb{R}^n \cong \mathbb{R}^n$. Its base consists of vectors $\frac{\partial}{\partial t^1}|_t, \dots, \frac{\partial}{\partial t^n}|_t$. Any mapping $D \ni t \mapsto \frac{\partial}{\partial t^j}|_t \in T_t D$ is a smooth vector field on D (see [4]). On the other hand on $TD \subset T[0, 1]^n = [0, 1]^n \times \mathbb{R}^n$ there exists the standard uniform structure inherited from the space $[0, 1]^n \times \mathbb{R}^n$ (given by coordinates of the mapping $TD \ni \sum_{j=1}^n v^j \frac{\partial}{\partial t^j}|_t \mapsto (t^1, \dots, t^n, v^1, \dots, v^n) \in [0, 1]^n \times \mathbb{R}^n$) (see [3]). We have:

Proposition 2. *Any vector field $\frac{\partial}{\partial t^i} : D \rightarrow TD$ is a uniform mapping with respect to standard uniform structures on D and TD .*

Proof. Let for given $\varepsilon > 0$

$$V_\varepsilon := \left\{ \left(\sum_{j=1}^n v^j \frac{\partial}{\partial t^j}|_t, \sum_{j=1}^n w^j \frac{\partial}{\partial t^j}|_s \right) \in \right.$$

$$\left. TD \times TD : |t^j - s^j| < \varepsilon \quad \wedge \quad |v^j - w^j| < \varepsilon, \quad j = 1, 2, \dots, n \right\}.$$

A family $\{V_\varepsilon\}_{\varepsilon \in (0; +\infty)}$ is a base of the standard uniform structure on TD , and a family $\{W_\varepsilon\}_{\varepsilon \in (0; +\infty)}$, where

$$W_\varepsilon := \{(t, s) \in D \times D : |t^j - s^j| < \varepsilon, \quad j = 1, 2, \dots, n\}$$

is a base of the standard uniform structure on D . It is clear that if for any $j \in \mathbb{N}$ $(t, s) = (t^1, \dots, t^n, s^1, \dots, s^n) \in W_\varepsilon$, then $(\frac{\partial}{\partial t^j}|_t, \frac{\partial}{\partial s^j}|_s) \in V_\varepsilon$. So we have the thesis (see [3] and [4]). \square

Proposition 3. *Let \mathcal{G} be a family of generators of a differential structure \mathcal{C} on a set M . If $\varphi : D \rightarrow M$ is such a smooth n -dimensional generalized cube on M that the tangent mapping $T\varphi : TD \rightarrow TM$ is uniform with respect to the uniform structure given on TM by the family \mathcal{TG}_0 , then for any function $\alpha \in \mathcal{G}$ the point 1-form $\varphi^*d\alpha$ smooth on TD can be extended to the 1-form $\widetilde{\varphi^*d\alpha}$ continuous on $[0, 1]^n$.*

Proof. The form $\varphi^*d\alpha$ is represented by the formula

$$\varphi^*d\alpha = \sum_{i=1}^n \frac{\partial(\alpha \circ \varphi)(t)}{\partial t^i} dt^i, \quad t \in D,$$

where for any $i = 1, 2, \dots, n$

$$\frac{\partial(\alpha \circ \varphi)}{\partial t^i}(t) = (\varphi^*d\alpha) \frac{\partial}{\partial t^i}|_t = d\alpha(T\varphi(\frac{\partial}{\partial t^i}|_t)).$$

Since the mapping $T\varphi$ is uniform on TD , the vector field $\frac{\partial}{\partial t^i}$ is a uniform mapping with respect to D in TD , and $d\alpha$ is one of functions of the family \mathcal{TG}_0 , so $\frac{\partial(\alpha \circ \varphi)}{\partial t^i}$ is a uniform function on D and it can be extended to the continuous function $\widetilde{\frac{\partial(\alpha \circ \varphi)}{\partial t^i}}$ on $[0, 1]^n$ (see [3] and [4]). So $\varphi^*d\alpha$ might be extended to the continuous 1-form $\widetilde{\varphi^*d\alpha} = \sum_{i=1}^n \widetilde{\frac{\partial(\alpha \circ \varphi)}{\partial t^i}}(t) dt^i$. \square

Conclusion 1. *Let the assumptions of the previous proposition be satisfied. Then for all $\alpha^1, \dots, \alpha^k \in \mathcal{G}$ the point k -form $\varphi^*(d\alpha^1 \wedge \dots \wedge d\alpha^k)$ can be extended to the k -form $\varphi^*(d\alpha^1 \wedge \dots \wedge d\alpha^k)$, which is continuous on $[0, 1]^n$.*

Proof. We have:

$$\varphi^*(d\alpha^1 \wedge \dots \wedge d\alpha^k) = (\varphi^*d\alpha^1) \wedge \dots \wedge (\varphi^*d\alpha^k).$$

So $\varphi^*(d\alpha^1 \wedge \dots \wedge d\alpha^k)$ could be extended to the k -form

$$\varphi^*(d\alpha^1 \wedge \dots \wedge d\alpha^k) := (\widetilde{\varphi^*d\alpha^1}) \wedge \dots \wedge (\widetilde{\varphi^*d\alpha^k}). \quad \square$$

Proposition 4. *Let the assumptions of Proposition 3 be satisfied. Let $k, m \in \mathbb{N}$, $\alpha_j^1, \dots, \alpha_j^k \in \mathcal{G}$ for $j = 1, 2, \dots, m$ and functions $\beta_1, \dots, \beta_m : M \rightarrow \mathbb{R}$ are uniform with respect to the uniform structure $\mathcal{U}_\mathcal{G}$ on M , then the point k -form $\eta := \varphi^*(\sum_{j=1}^m \beta_j d\alpha_j^1 \wedge \dots \wedge d\alpha_j^k)$ can be extended to the k -form $\tilde{\eta}$, which is continuous on $[0, 1]^n$.*

Proof. Since $\eta = \sum_{j=1}^m (\beta_j \circ \varphi) \varphi^*(d\alpha_j^1 \wedge \dots \wedge d\alpha_j^k)$ it is enough to take $\tilde{\eta} = \sum_{j=1}^m (\widetilde{\beta_j \cdot \varphi}) \cdot \varphi^*(d\alpha_j^1 \wedge \dots \wedge d\alpha_j^k)$ (see Conclusion 1). \square

Proposition 5. *Let η be a smooth point k -form on a differential space (M, \mathcal{C}) , and \mathcal{G} be a family of generators of the differential structure \mathcal{C} . Then for any $p \in M$ there exists a neighborhood U , a number $m \in \mathbb{N}$, functions $\alpha_j^1, \dots, \alpha_j^k \in \mathcal{G}$ for $j = 1, 2, \dots, m$ and functions $\beta_1, \dots, \beta_m \in \mathcal{C}$ uniform with respect to the uniform structure $\mathcal{U}_{\mathcal{G}}$ such that*

$$\eta|_U = \sum_{j=1}^m \beta_{j|U} (d\alpha_j^1)|_U \wedge \dots \wedge (d\alpha_j^k)|_U.$$

Proof. It follows from Theorem 1 and Remark 6 that there exists a neighborhood W_p of p such that η can be written in the form (4) on W_p . Let us fix any 1-1 map $J : \{1, 2, \dots, \frac{n!}{k!(n-k)!}\} \rightarrow \{(i_1, \dots, i_k) \in \mathbb{N}^k : 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$ and let $J(j) = (i_1(j), \dots, i_k(j))$. Then putting $m := \frac{n!}{k!(n-k)!}$, $\omega_j := \omega_{J(j)}$ and $\alpha_j^r := \alpha_{i_r(j)}$ for any $j \in \{1, 2, \dots, m\}$ and $1 \leq r \leq k$ we obtain that

$$\eta|_{W_p} = \sum_{j=1}^m \omega_{J(j)|W_p} (d\alpha_j^1 \wedge \dots \wedge (d\alpha_j^k))|_{W_p}, \quad (6)$$

where $\alpha_j^1, \dots, \alpha_j^k \in \mathcal{G}$ for $j = 1, 2, \dots, m$ and functions $\omega_1, \dots, \omega_m \in \mathcal{C}$ (the number m , functions $\alpha_j^1, \dots, \alpha_j^k$ and $\omega_1, \dots, \omega_m$ could depend on W_p). Recall now that $\omega_j = 0$ or ω_j is the superposition of the smooth map $\beta_0 \in \mathcal{G}^{n_0}$ which takes a value in an n_0 -dimensional interval U^0 and a smooth function (partial derivative $\frac{\partial^k \sigma}{\partial u_{j_1}^1 \dots \partial u_{j_k}^k}$) defined on \mathbb{R}^{n_0} (see proof of Theorem 1). Since $\frac{\partial^k \sigma}{\partial u_{j_1}^1 \dots \partial u_{j_k}^k}|_{U^0}$ is an uniform function on U^0 and coefficients of β^0 are uniform functions with respect to the uniform structure $\mathcal{U}_{\mathcal{G}}$ on M (as members of \mathcal{G}) we obtain that ω_j are uniform with respect to $\mathcal{U}_{\mathcal{G}}$ on the set $U := W_p$ as the superposition of uniform mappings. The thesis follows now from (6). \square

Theorem 2. *Let η be a smooth point k -form on a differential space (M, \mathcal{C}) , and \mathcal{G} be a family of generators of the differential structure \mathcal{C} . Let us assume that:*
(i) *there exists an open finite covering $\{U_i\}_{i \in I}$ ($I = \{1, \dots, q\}$, $q \in \mathbb{N}$) of the space M , such that for any $i \in I$ there exist a number $m_i \in \mathbb{N}$, functions $\alpha_{i,j}^1, \dots, \alpha_{i,j}^k \in \mathcal{G}$ for $j = 1, \dots, m_i$ and functions $\beta_{i,1}, \dots, \beta_{i,m_i} \in \mathcal{C}$ uniform with respect to the uniform structure $\mathcal{U}_{\mathcal{G}}$ such that $\eta|_{U_i} = \sum_{j=1}^{m_i} \beta_{i,j|U_i} (d\alpha_{i,j}^1)|_{U_i} \wedge \dots \wedge (d\alpha_{i,j}^k)|_{U_i}$;*
(ii) *there exist a smooth partition of unity $\{\gamma_i\}_{i \in I}$ subordinated to the covering $\{U_i\}_{i \in I}$ such that any function γ_i , $i \in I$, is uniform with respect to $\mathcal{U}_{\mathcal{G}}$. Then for*

any smooth n -dimensional generalized cube $\varphi : D \rightarrow M$ such that $T\varphi$ is uniform with respect to $\mathcal{T}\mathcal{G}_0$, k -form $\varphi^*\eta$ is smooth on D and could be extended to the k -form $\widetilde{\varphi^*\eta}$ continuous on $[0, 1]^n$.

Proof. We have $\eta = \sum_{i \in I} \gamma_i \eta$, and $\varphi^*\eta = \sum_{i \in I} \varphi^*(\gamma_i \eta)$. But for any $i \in I$ $\text{supp} \gamma_i \subset U_i$ so we have

$$\gamma_i \eta = \sum_{j=1}^{m_i} \gamma_i \cdot \beta_{i,j} d\alpha_{i,j}^1 \wedge \dots \wedge d\alpha_{i,j}^k.$$

Therefore

$$\varphi^*(\gamma_i \eta) = \sum_{j=1}^{m_i} (\gamma_i \circ \varphi) \varphi^*[\beta_{i,j} d\alpha_{i,j}^1 \wedge \dots \wedge d\alpha_{i,j}^k]$$

and the form could be extended on $[0, 1]^n$ to the k -form

$$\widetilde{\varphi^*(\gamma_i \eta)} := \sum_{j=1}^{m_i} (\widetilde{\gamma_i \circ \varphi}) \varphi^*[\widetilde{\beta_{i,j} d\alpha_{i,j}^1 \wedge \dots \wedge d\alpha_{i,j}^k}]$$

(see Proposition 4). Hence $\varphi^*\eta$ could be extended to the k -form

$$\widetilde{\varphi^*\eta} = \sum_{i \in I} \widetilde{\varphi^*(\gamma_i \eta)}. \quad \square$$

If a point k -form η and a smooth generalized n -dimensional cube φ on a differential space (M, \mathcal{C}) satisfy the assumptions of Theorem 2, then there exists $\int_\varphi \eta$ (of course it is equal to zero for $n \neq k$). Similarly if Ψ is a smooth n -dimensional chain on (M, \mathcal{C}) and each cube $\varphi \in \text{supp} \Psi$ together with a point k -form η satisfy the assumptions of Theorem 2 then there exists $\int_\Psi \eta$.

For the further development of the theory of integration we need a generalization of the operation of boundary. The operation prescribes to any n -dimensional cube φ an $(n-1)$ -dimensional chain $\partial\varphi$. It is connected with Stokes theorem and justifies the concept of chain. Standardly we can extend this operation to the set of chains. For any n -dimensional chain $\Phi = \sum_{j=1}^n c_j \varphi_j$ we have $\partial\Phi = \sum_{j=1}^n c_j \partial\varphi_j$.

To introduce the operation of boundary on generalized cubes and chains in differential spaces we need some new concepts.

Definition 12. Let \mathcal{G} be a family of generators of a differential structure \mathcal{C} on a set M . We call a smooth generalized n -dimensional cube $\varphi : D \rightarrow M$ on (M, \mathcal{C}) *smoothly extendable with respect to \mathcal{G}* if the tangent mapping $T\varphi$ is uniform with respect to the uniform structure given on TM by the family $\mathcal{T}\mathcal{G}_0$ (see [2]) and its

continuous extension $\widetilde{T\varphi} : [0; 1]^n \times \mathbb{R}^n \rightarrow \text{compl}_{\mathcal{T}\mathcal{G}_0} TM$ is a smooth map with respect to differential structures $C^\infty([0; 1]^n \times \mathbb{R}^n)$ and $\text{compl}_{\mathcal{T}\mathcal{G}_0} \mathcal{TC}$ respectively. The chain $\Phi \in Ch_n(M)$ is *smoothly extendable with respect to \mathcal{G}* when every cube $\varphi \in \text{supp}\Phi$ is smoothly extendable with respect to \mathcal{G} . The smooth point k -form η on M is said to be *smoothly extendable with respect to \mathcal{G}* if there exists a smooth point k -form $\tilde{\eta}$ on $\text{compl}_{\mathcal{G}} M$, such that $\eta = \iota^*(\tilde{\eta})$, where $\iota : M \rightarrow \text{compl}_{\mathcal{G}} M$ is the natural embedding (for a point $p \in M$ we put its filter of neighborhoods $\iota(p) = \mathcal{F}_p$ – see [2]).

Let $\pi : TM \rightarrow M$ be the canonical projection. Then for each set of generators \mathcal{G} of the differential structure \mathcal{C} the mapping π is uniform with respect to uniform structures $\mathcal{U}_{\mathcal{T}\mathcal{G}_0}$ and $\mathcal{U}_{\mathcal{G}}$ (from definitions of π and $\mathcal{U}_{\mathcal{T}\mathcal{G}_0}$). Moreover for each differential space (N, \mathcal{D}) and each $F : (N, \mathcal{D}) \rightarrow (M, \mathcal{C})$ we have $\pi \circ TF = F$. So the fact that a smooth generalized n -dimensional cube $\varphi : \mathcal{D} \rightarrow M$ is smoothly extendable implies that φ is a uniform mapping and its extension $\tilde{\varphi} : [0, 1]^n \rightarrow \text{compl}_{\mathcal{G}} M$ is smooth, i.e. it is a smooth n -dimensional cube on $\text{compl}_{\mathcal{G}} M$.

Definition 13. Let $\varphi : D \rightarrow M$ be a smooth generalized n -dimensional cube on a differential space (M, \mathcal{C}) , smoothly extendable with respect to some family of generators \mathcal{G} of the structure \mathcal{C} . We define *the boundary of φ with respect to \mathcal{G}* as the n -dimensional smooth chain $\partial_{\mathcal{G}}\varphi$ on $\text{compl}_{\mathcal{G}} M$ given by the formula:

$$\partial_{\mathcal{G}}\varphi = \sum_{i=1}^n \sum_{\alpha=0,1} (-1)^{i+\alpha} \tilde{\varphi}_{(i,\alpha)},$$

where $\tilde{\varphi}_{(i,\alpha)} = \tilde{\varphi} \circ (I_{(i,\alpha)}^n)$ is *the (i, α) -face* of the cube $\tilde{\varphi}$,

$$I_{(i,0)}^n(x) = (x^1, \dots, x^{i-1}, 0, x^i, \dots, x^{n-1}),$$

$$I_{(i,1)}^n(x) = (x^1, \dots, x^{i-1}, 1, x^i, \dots, x^{n-1})$$

and $\tilde{\varphi}$ is the smooth extension of φ on $\text{compl}_{\mathcal{G}} M$.

It is easy to see that $\partial_{\mathcal{G}}\varphi = \partial_{\tilde{\mathcal{G}}}\tilde{\varphi}$, where $\tilde{\mathcal{G}}$ is the family of extensions of elements of the set \mathcal{G} on $\text{compl}_{\mathcal{G}} M$.

Definition 14. The boundary of a generalized n -dimensional chain $\Phi = \sum_{j=1}^m c_j \varphi_j$ on the differential space (M, \mathcal{C}) smoothly extendable with respect to a family of generators \mathcal{G} of the structure \mathcal{C} we call the $(n - 1)$ -dimensional chain:

$$\partial_{\mathcal{G}}\Phi := \sum_{j=1}^m c_j \partial_{\mathcal{G}}\varphi_j,$$

where $\{\varphi_1, \dots, \varphi_m\} = \text{supp}\Phi$.

For formulating the analogue of Stokes theorem we need a mapping that will be an analogue of the exterior derivative. That mapping should be a generalization of the exterior derivative on manifolds and it should linearly maps k -forms to $(k + 1)$ -forms. It should also commute with pullback of forms. Next example shows that we can not give such mapping uniquely even on smooth extendable forms.

Example 3. Let $M = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$, and $\mathcal{C} = C^\infty(\mathbb{R}^2)_M$. Then the zero 1-form on (M, \mathcal{C}) could be extended to the zero 1-form on \mathbb{R}^2 and to the form $\eta = xdy - ydy$. We have: $d\eta = 2dx \wedge dy$. Moreover if by $\iota : M \rightarrow \mathbb{R}^2$ we denote the natural embedding, then $\iota^*(dx \wedge dy) = dx \wedge dy \neq 0$, because $dx \wedge dy|_{T_0 M} \neq 0$, while $\iota^*(0) = 0$. \square

If \mathcal{G} is a family of bounded generators of a differential structure \mathcal{C} on a set M (we can assume that if $g \in \mathcal{G}$, then $|g| \leq 1$), then the completion $\text{compl}_{\mathcal{G}} M$ is a differential compactification $\text{compt}_{\mathcal{G}} M$ (see [2]).

Proposition 6. *Let the previous assumption for \mathcal{G} be satisfied. Then any point k -form η that is smoothly extendable with respect to \mathcal{G} can be extended to a smooth k -form η_1 on $[-1, 1]^{\mathcal{G}}$.*

Proof. Let $\tilde{\eta}$ be such a smooth point k -form on $\text{compl}_{\mathcal{G}} M$ that $\eta = i^* \tilde{\eta}$, where i is the natural embedding M into $\text{compl}_{\mathcal{G}} M$. Let $\tilde{\mathcal{G}}$ be the set of all continuous extensions of elements of \mathcal{G} onto $\text{compl}_{\mathcal{G}} M$. Since $\tilde{\mathcal{G}}$ is a family of generators of the differential structure $\text{compl}_{\mathcal{G}} \mathcal{C} = C^\infty([-1, 1]^{\mathcal{G}})_{\text{compl}_{\mathcal{G}} M}$ we can choose for any $p \in \text{compl}_{\mathcal{G}} M$ a neighborhood W_p such that (4) holds for the form $\tilde{\eta}$ and for some $\tilde{\alpha}_{i_j} \in \tilde{\mathcal{G}}$. If we take a set V_p open in $[-1, 1]^{\mathcal{G}}$ and such that $W_p = \text{compl}_{\mathcal{G}} M \cap V_p$ then the same formula defines the point k -form $\tilde{\eta}_p$ on V_p . But the space $\text{compl}_{\mathcal{G}} M$ is compact and therefore there exists $n \in \mathbb{N}$ and $p_1, \dots, p_n \in \text{compl}_{\mathcal{G}} M$ such that $\text{compl}_{\mathcal{G}} M \subset \bigcup_{j=1}^n V_{p_j}$. Putting $V_0 := [-1, 1]^{\mathcal{G}} \setminus \text{compl}_{\mathcal{G}} M$, $V_j := V_{p_j}$ and $\tilde{\eta}_j := \tilde{\eta}_{p_j}$ for $j = 1, \dots, n$ we obtain the open finite covering $\{V_0, V_1, \dots, V_n\}$ of $[-1, 1]^{\mathcal{G}}$. Let $\{\gamma_0, \gamma_1, \dots, \gamma_n\}$ be a smooth partition of unity subordinated to the covering. Than any $\gamma_j \tilde{\eta}_j$, $j = 1, \dots, n$, can be in a standard manner extend to $[-1, 1]^{\mathcal{G}}$ (we take $(\gamma_j \tilde{\eta}_j)|_{[-1, 1]^{\mathcal{G}} \setminus W_j} = 0$). Taking $\eta_1 := \sum_{j=1}^n \gamma_j \tilde{\eta}_j$ we obtain the k -form on $[-1, 1]^{\mathcal{G}}$ which fulfils the conditions of the proposition. \square

On the differential space $([-1, 1]^{\mathcal{G}}, C^\infty([-1, 1]^{\mathcal{G}}))$ there is the well definite operator of exterior derivative d which assigns to any smooth point k -form ω the smooth $(k + 1)$ -form $d\omega$ given by the formula

$$(d\omega)(v_1, \dots, v_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \dots, \hat{X}_i, \dots, X_{k+1}))(p) +$$

$$+ \dots \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1})(p), \quad (7)$$

where $v_1, \dots, v_{k+1} \in T_p([-1, 1]^{\mathcal{G}})$, X_1, \dots, X_{k+1} are vector fields on $[-1, 1]^{\mathcal{G}}$, such that $X_j(p) = v_j$ for $j = 1, 2, \dots, n$ (such a vector fields exist on a space which is a Cartesian product of intervals; we can take $X_j = \text{const} = v_j \in T[-1, 1]^{\mathcal{G}} \cong \mathbb{R}^{\mathcal{G}}$ – see [1], [2], [3] or [4]) and “ \wedge ” means that the variable disappears in that expression.

For any smooth mapping $F : [0; 1]^n \rightarrow [-1, 1]^{\mathcal{G}}$ and any smooth point k -form ω smoothly extendable with respect to \mathcal{G} operator d fulfils the equality: $d_1(F^*\chi) = F^*d\chi$, where on the left hand side there is “the classical” operator d_1 of the exterior derivative on $[0; 1]^n$. It is the result of (7), definitions and properties of vector fields. Now we can get the analogue of Stokes theorem.

Theorem 3. *If a smooth point k -form η and a smooth generalized n -dimensional chain Φ are smoothly extendable with respect to a family \mathcal{G} of bounded generators of a differential structure \mathcal{C} on a set M , then*

$$\int_{\partial_{\mathcal{G}}\Phi} \eta = \int_{\Phi} d\tilde{\eta}, \quad (8)$$

where $\tilde{\eta}$ is an arbitrary smooth extension of the k -form η on the differential space $([-1, 1]^{\mathcal{G}}, C^\infty([-1, 1]^{\mathcal{G}}))$ and d is the operator of exterior derivative given by (7).

Proof. It is enough to show equality (8) for any n -dimensional cube φ on (M, \mathcal{C}) smoothly extendable with respect to \mathcal{G} . We have

$$\int_{\partial_{\mathcal{G}}\varphi} \eta = \int_{\partial([0; 1]^n)} \widetilde{\varphi^*\eta} = \int_{\partial([0; 1]^n)} \tilde{\varphi}^*\tilde{\eta} = \int_{[0; 1]^n} d_1(\tilde{\varphi}^*\tilde{\eta}) = \int_{[0; 1]^n} \tilde{\varphi}^*d\tilde{\eta} = \int_{\varphi} d\tilde{\eta},$$

where d_1 is the operator of exterior derivative on $[0; 1]^n$ and $\tilde{\varphi}$ is the extension of φ on $[0; 1]^n$. \square

References

- [1] D.Dziewa-Dawidczyk, *Integration on differential spaces (doctor thesis - in polish)*, Warsaw University of Technology, Warsaw 2010.
- [2] D.Dziewa-Dawidczyk, Z. Pasternak-Winiarski, *On differential completions and compactifications of a differential space*, Demonstratio Math. 45(4) (2012), 975-992; arXiv:1103.3597, 2011.

- [3] D. Dziewa-Dawidczyk, Z. Pasternak-Winiarski, *Uniform structures on differential spaces*, arXiv:1103.2799, 2011.
- [4] D. Dziewa-Dawidczyk, Z. Pasternak - Winiarski, *Differential structures on natural bundles connected with a differential space*, in "Singularities and Symplectic Geometry VII" Singularity Theory Seminar (2007), S. Janeczko (ed), Faculty of Mathematics and Information Science, Warsaw University of Technology.
- [5] R. Engelking, *General topology*, PWN, Warsaw 1975 (in polish).
- [6] Z. Pasternak - Winiarski, *Group differential structures and their basic properties (doctor thesis - in polish)*, Warsaw University of Technology, Warsaw 1981.
- [7] W. Sasin, *On some exterior algebra of differential forms over a differential space*, Demonstratio Math., 19 (1986), 1063-1075.
- [8] R. Sikorski, *Introduction to differential geometry*, PWN, Warsaw 1972 (in polish).
- [9] W. Waliszewski, *Regular and coregular mappings of differential space*, Annales Polonici Mathematici XXX, 1975.